ON THE MULTIPLE q-GENOCCHI AND EULER NUMBERS

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ABSTRACT. The purpose of this paper is to present a systemic study of some families of multiple q-Genocchi and Euler numbers by using multivariate q-Volkenborn integral (= p-adic q-integral) on $\mathbb{Z}_p$. From the studies of those q-Genocchi numbers and polynomials of higher order we derive some interesting identities related to q-Genocchi numbers and polynomials of higher order.

§1. Introduction

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$ and let q be regarded as either a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume $|1-q|_p < p^{-\frac{1}{p-1}}$, which implies that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Here, $|\cdot|_p$ is the p-adic absolute value in $\mathbb{C}_p$ with $|p|_p = \frac{1}{p}$. The q-basic natural number are defined by $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$, ($n \in \mathbb{N}$), and q-factorial are also defined as $[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$. In this paper we use the notation of Gaussian binomial coefficient as follows:

$$(1)\qquad \binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q!}.$$

Note that $\lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{n!}$. The Gaussian coefficient satisfies the following recursion formula:

$$(2)\qquad \binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k}\binom{n}{k-1}_q + \binom{n}{k}_q, \text{ cf. [12].}$$

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From thus recursion formula we derive

$$\binom{n}{k}_q = \sum_{d_0+\cdots+d_k=n-k, d_i\in\mathbb{N}} q^{d_1+2d_2+\cdots+kd_k}, \text{ cf.[1, 2, 12, 13, 14].}$$

The q-binomial formulae are known as

$$(b;q)_n = \prod_{i=1}^{n}\left(1-bq^{i-1}\right) = \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}(-1)^k b^k,$$

and

$$\text{(3)} \qquad \frac{1}{(b;q)_n} = \prod_{i=1}^{n}\left(1-bq^{i-1}\right)^{-1} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}_q b^k, \text{ cf.[12].}$$

In this paper we use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

Hence, $\lim_{q\to 1}[x]_q = 1$, for any x with $|x|_p \leq 1$ in the present p-adic case, cf.[1-18].

For d a fixed positive integer with $(p,d)=1$, let

$$X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \ X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a+dp\mathbb{Z}_p),$$
$$a+dp^N\mathbb{Z}_p = \{x\in X | x\equiv a \pmod{dp^n}\},$$

where $a\in\mathbb{Z}$ lies in $0\leq a< dp^N$. In [9], we note that

$$\mu_{-q}(a+dp^N\mathbb{Z}_p) = (1+q)\frac{(-1)^a q^a}{1+q^{dp^N}} = \frac{(-q)^a}{[dp^N]_{-q}},$$

is distribution on X for $q\in\mathbb{C}_p$ with $|1-q|_p<1$. We say that f is a uniformly differentiable function at a point $a\in\mathbb{Z}_p$ and denote this property by $f\in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = \dfrac{f(x)-f(y)}{x-y}$ have a limit $l=f'(a)$ as $(x,y)\to(a,a)$. For $f\in UD(\mathbb{Z}_p)$, this distribution yields an integral as follows:

$$I_{-q} = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \int_X f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{[dp^N]_{-q}}\sum_{x=0}^{dp^N-1} f(x)(-q)^x,$$

which has a sense as we see readily that the limit is convergent (see [9, 10, 14, 15]). Let $q = 1$. Then we have the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1} = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ cf.[3, 6, 7, 8, 9, 13, 14]}.$$

For any positive integer N, we set

$$\mu_q(a + lp^N\mathbb{Z}_p) = \frac{q^a}{[lp^N]_q}, \text{ cf. [5, 9, 15, 16, 17, 18]},$$

and this can be extended to a distribution on X. This distribution yields p-adic bosonic q-integral as follows (see [12, 17, 18]):

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \int_X f(x)\, d\mu_q(x),$$

where $f \in UD(\mathbb{Z}_p) =$ the space of uniformly differentiable function on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$.

In view of notation, I_{-1} can be written symbolically as $I_{-1}(f) = \lim_{q\to-1} I_q(f)$, cf.[9]. For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$. Then we have

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \text{ see [9]}. \tag{4}$$

For any complex number z, it is well known that the familiar Euler polynomials $E_n(z)$ are defined by means of the following generating function:

$$F(z, t) = \frac{2}{e^t + 1} e^{zt} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}, \text{ for } |t| < \pi, \text{ cf.[13,14]}.$$

We note that, by substituting $z = 0$, $E_n(0) = E_n$ is the familiar n-th Euler number defined by

$$F(t) = F(0, t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ cf.[12]}.$$

The Genocchi numbers G_n are defined by the generating function

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi).$$

It satisfies $G_1 = 1, G_3 = G_5 = \cdots = G_{2k+1} = 0$, and even coefficients are given by

$$G_n = 2(1 - 2^n)B_n = 2nE_{2n-1}(0),$$

where B_n are Bernoulli numbers and $E_n(x)$ are Euler polynomials. By meaning of the generalization of E_n, Frobenius-Euler numbers and polynomials are also defined by

$$\frac{1-u}{e^t-u}=\sum_{n=0}^{\infty}H_n(u)\frac{t^n}{n!},\ \text{and}\ \frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(u,x)\frac{t^n}{n!},\ \text{for}\ u\in\mathbb{C},\ \text{cf.[12, 14]}.$$

Over five decades ago, Carlitz [1, 2] defined q-extension of Frobenius-Euler numbers and polynomials and proved properties analogous to those satisfied $H_n(u)$ and $H_n(u,x)$. In previous my paper [6, 7, 8] the author defined the q-extension of ordinary Euler and polynomials and proved properties analogous to those satisfied E_n and $E_n(x)$. In [6] author also constructed the q-Euler numbers and polynomials of higher order and gave some interesting formulae related to Euler numbers and polynomials of higher order . The purpose of this paper is to present a systemic study of some families of multiple q-Genocchi and Euler numbers by using multivariate q-Volkenborn integral (= p-adic q-integral) on $\mathbb{Z}_p$. From the studies of these q-Genocchi numbers and polynomials of higher order we derive some interesting identities related to q-Genocchi numbers and polynomials of higher order.

§2. Preliminaries / q-Euler polynomials

In this section we assume that $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-\frac{1}{p-1}}$. Let $f_1(x)$ be translation with $f_1(x)=f(x+1)$. From (4) we can derive

$$I_{-1}(f_1)=I_{-1}(f)+2f(0).$$

If we take $f(x)=e^{(x+y)t}$, then we have Euler polynomials from the integral equation of $I_{-1}(f)$ as follows:

$$\int_{\mathbb{Z}_p}e^{(x+y)t}d\mu_{-1}(y)=e^{xt}\frac{2}{e^t+1}=\sum_{n=0}^{\infty}\frac{E_n(x)t^n}{n!}.$$

That is,

$$\int_{\mathbb{Z}_p}y^n d\mu_{-1}(y)=E_n,\ \text{and}\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y)=E_n(x).$$

Now we consider the following multivariate p-adic fermionic integral on $\mathbb{Z}_p$ as follows:
(5)

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r+x)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!},$$

where $E_n^r(x)$ are the Euler polynomials of order r.

From (5) we note that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x). \tag{6}$$

In view of (6) we can define the q-extension of Euler polynomials of higher order. For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, let us consider the extended higher order q-Euler polynomials as follows:

$$E_{m,q}^{(h,k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_q^m q^{\sum_{j=1}^k x_j (h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r), \text{ see [6] }.$$

From this definition we can derive

$$E_{m,q}^{(h,k)}(x) = [2]_q^k \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{xj} \frac{1}{(-q^{j+h}; q^{-1})_k}, \text{ see [6] }. \tag{7}$$

It is easy to see that

$$(-q^{j+k-1}; q^{-1})_k = \prod_{l=0}^{k-1} (1 + q^l q^j) = (-q^j; q)_k$$

In the special case $h = k-1$, we can easily see that

$$\begin{aligned}
E_{m,q}^{(k-1.k)}(x) &= [2]_q^k \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{xj} \frac{1}{(-q^{j+k-1}; q^{-1})_k} \\
&= [2]_q^k \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{xj} \frac{1}{(-q^j; q)_k} \\
&= [2]_q^k \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{xj} \sum_{n=0}^{\infty} \binom{k+n-1}{n}_q (-q^j)^n \\
&= [2]_q^k \sum_{n=0}^{\infty} \binom{k+n-1}{n}_q (-1)^n [n+x]_q^m .
\end{aligned} \tag{8}$$

Let $F^k(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(k-1,k)}(x)$ be the generating function of $E_{n,q}^{k-1,k}(x)$. From (8) we note that

$$\begin{aligned}
F_q^k(t,x) &= \sum_{m=0}^{\infty} E_{m,q}^{(k-1,k)}(x) \frac{t^m}{m!} = [2]_q^k \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{k+n-1}{n}_q (-1)^n [n+x]_q^m \frac{t^m}{m!} \\
&= [2]_q^k \sum_{n=0}^{\infty} \binom{k+n-1}{n}_q (-1)^n e^{[n+x]_q} .
\end{aligned}$$

Remark. For $w \in \mathbb{C}_p$ with $|1-w|<1$, we have

$$I_{-1}(w^x e^{tx}) = \int_{\mathbb{Z}_p} w^x e^{tx} d\mu_{-1}(x) = \frac{2}{we^t+1} = \sum_{n=0}^{\infty} E_n(w)\frac{t^n}{n!}, \text{ see [3, 9, 11, 12, 15]} .$$

Thus, we note that $\int_{\mathbb{Z}_p} w^x x^n d\mu_{-1}(x) = E_n(w)$ and $E_n(w) = \frac{2}{w+1}H_n(-w^{-1})$, where $H_n(-w^{-1})$ are Frobenius-Euler numbers.

In the previous paper [11], the q-extension of $E_n(w)$(= twisted q-Euler numbers) are studied as follows:

$$I_{-q}(w^x e^{t[x]_q}) = \int_{\mathbb{Z}_p} w^x e^{t[x]} d\mu_{-q}(x) = \sum_{n=0}^{\infty} E_{n,q}(w)\frac{t^n}{n!}. \tag{9}$$

From (9) we note that

$$E_{n,q}(w) = \int_{\mathbb{Z}_p} w^x [x]_q^n d\mu_{-q}(x) = \frac{[2]_q}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j \frac{1}{1+q^{j+1}w}, \text{ see [11]} .$$

By the exactly same method of Eq.(7), we can also derive the multiple twisted q-Euler numbers as follows:

$$\begin{aligned} &E_{m,q}^{(h,k)}(w,x) \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{j=1}^k x_j}[x_1+\cdots+x_k+x]_q^m q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k). \end{aligned} \tag{10}$$

From (10) we can easily derive

$$E_{m,q}^{(h,k)}(w,x) = \frac{[2]_q^k}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}(-q^x)^l \frac{1}{(-wq^{h+l};q^{-1})}.$$

For $h=k-1$, we have

$$\begin{aligned} E_{m,q}^{(k-1,k)}(w,x) &= \frac{[2]_q^k}{(1-q)^m}\sum_{l=0}^{m}\binom{m}{l}(-q^x)^l \frac{1}{(-wq^l;q)_k} \\ &= [2]_q^k \sum_{n=0}^{\infty}\binom{n+k-1}{n}_q (-w)^n [n+x]_q^m. \end{aligned} \tag{11}$$

Let $F_q^k(w,x) = \sum_{m=0}^{\infty} E_{m,q}^{(k-1,k)}(x)\frac{t^m}{m!}$. From (11), we can easily derive

$$F_q^k(w,x) = [2]_q^k \sum_{n=0}^{\infty}\binom{n+k-1}{n}_q (-w)^n e^{[n+x]_q t}.$$

Remark. When we consider those q-Euler numbers and polynomials in complex number field, we assume that $q \in \mathbb{C}$ with $|q| < 1$.

§3. Genocchi and q-Genocchi numbers

From (4) we note that

$$t\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \tag{12}$$

Thus, we have

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1}\frac{t^n}{n!}. \tag{13}$$

By (13) we easily see that

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \text{ and } \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1} d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1},$$

where $G_n(x)$ are Genocchi polynomials (see [8]).

From the multivariate p-adic fermionic integral on $\mathbb{Z}_p$ we can also derive the Genocchi numbers of order r as follows:

(14)

$$t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \left(\frac{2t}{e^t+1}\right)^r = \sum_{n=0}^{\infty} G_n^{(r)} \frac{t^n}{n!}, \ r \in \mathbb{N},$$

where $G_n^{(r)}$ are the Genocchi numbers of order r.

From (14) we note that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \frac{(r+n)_r t^{n+r}}{(n+r)!} = \sum_{n=0}^{\infty} G_n^{(r)} \frac{t^n}{n!}, \tag{15}$$

where $(x)_r = x(x-1)\cdots(x-r+1)$. By (14) and (15), we easily see that

(16)

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \frac{1}{r!\binom{n+r}{r}} G_{n+r}^{(r)}, \text{ where } n \in \mathbb{N}\cup\{0\},$$

and $G_0^{(r)} = G_1^{(r)} = \cdots = G_{r-1}^{(r)} = 0$. Thus, we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{N} \cup \{0\}$, $r \in \mathbb{N}$, we have*

$$G_{n+r}^{(r)} = (n+r)_r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$
$$= \binom{n+r}{r} r! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r),$$

where $(x)_r = x(x-1)\cdots(x-r+1)$.

Recently we constructed the q-extension of Genocchi numbers as follows:

$$t \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q}(x) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \text{ see [8] .} \tag{17}$$

Thus, we note that

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) = G_{n,q} = n \frac{[2]_q}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{1+q^{l+1}}, \text{ see [8] .}$$

In view of (14) we can define the q-extension of Genocchi numbers of higher order. For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, let us consider the extended higher order q-Genocchi numbers as follows:

$$\sum_{n=0}^{\infty} G_{n,q}^{(h,k)} \frac{t^n}{n!} = t^k \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1+\cdots+x_k]_q t} q^{\sum_{j=1}^k x_j(h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$
$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^k x_j(h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \frac{t^{n+k}}{n!}$$
$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^k x_j(h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \frac{(n+k)_k t^{n+k}}{(n+k)!}.$$

Thus, we have

$$G_{n+k,q}^{(h,k)} = k! \binom{n+k}{k} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^k x_j(h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$
$$= k! \binom{n+k}{k} \frac{[2]_q^k}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{(-q^{h+l}; q^{-1})_k},$$

and

$$G_{0,q}^{(h,k)} = G_{1,q}^{(h,k)} = \cdots = G_{k-1,q}^{(h,k)} = 0.$$

If we take $h = k-1$, then we have

$$\begin{aligned} G_{n+k,q}^{(k-1,k)} &= k!\binom{n+k}{k}\frac{[2]_q^k}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{(-q^l;q)_k} \\ &= k!\binom{n+k}{k}\frac{[2]_q^k}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-q^l)^m \\ &= k!\binom{n+k}{k}[2]_q^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-1)^m[m]_q^n. \end{aligned}$$

Therefore we obtain the following theorem.

Theorem 2. *For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, we have*

$$G_{n+k,q}^{(h,k)} = k!\binom{n+k}{k}\frac{[2]_q^k}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{(-q^{h+l};q^{-1})_k},$$

and

$$G_{n+k,q}^{(k-1,k)} = k!\binom{n+k}{k}[2]_q^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-1)^m[m]_q^n.$$

Let

$$h_q^k(t) = \sum_{n=0}^{\infty}G_{n,q}^{(k-1,k)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}G_{n+k,q}^{(k-1,k)}\frac{t^{n+k}}{(n+k)!}, \tag{18}$$

because $G_{0,q}^{(k-1,k)} = \cdots = G_{k-1,q}^{(k-1,k)} = 0$. By (18) and Theorem 2, we see that

$$\begin{aligned} h_q^k(t) &= \sum_{n=0}^{\infty}G_{n,q}^{(k-1,k)}\frac{t^n}{n!} = [2]_q^k t^k\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-1)^m[m]_q^n\frac{t^m}{m!} \\ &= [2]_q^k t^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-1)^m e^{[m]_q t}. \end{aligned}$$

Remark. For $w \in \mathbb{C}_p$ with $|1-w|_p < 1$, we can also define w-Genocchi numbers (= twisted Genocchi numbers) as follows:

$$t\int_{\mathbb{Z}_p}w^x e^{xt}d\mu_{-1}(x) = \frac{2t}{we^t+1} = \sum_{n=0}^{\infty}G_{n,w}\frac{t^n}{n!},\ \text{cf.}[3,\ 11,\ 13]\ .$$

From this we note that $\lim_{w\to 1}G_{n,w} = G_n$. The q-extension of $G_{n,w}$ can be also defined by

$$t\int_{\mathbb{Z}_p}w^x e^{[x]_q t}d\mu_{-q}(x) = \sum_{n=0}^{\infty}G_{n,q,w}\frac{t^n}{n!},\ \text{cf. }[3,\ 8,\ 11,\ 13]\ . \tag{19}$$

By (19) we easily see that

$$G_{n,q,w} = n\frac{[2]_q}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{(-1)^l}{1+q^{l+1}w}, \text{ cf.[11] }.$$

From this we also note that $\lim_{w\to 1} G_{n,q,w} = G_{n,q}$.

Now we consider the extended (q,w)-Genocchi numbers by using multivariate p-adic fermionic integral on $\mathbb{Z}_p$. For $h \in \mathbb{Z}, k \in \mathbb{N}$, $w \in \mathbb{C}_p$ with $|1-w|_p < 1$, we define $G^{(h,k)}_{n,q,w}$ as follows:

$$(20)\quad \begin{aligned}&\sum_{n=0}^{\infty} G^{(h,k)}_{n,q,w}\frac{t^n}{n!}\\ &= t^k \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k}e^{[x_1+\cdots+x_k]_q t}q^{\sum_{j=1}^k x_j(h-j)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).\end{aligned}$$

From (20) we can derive

$$G^{(h,k)}_{n+k,q,w} = k!\binom{n+k}{k}\frac{[2]_q^k}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{(-wq^{h+l};q^{-1})_k},$$

and

$$G^{(k-1,k)}_{n+k,q,w} = k!\binom{n+k}{k}[2]_q^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-w)^m[n]_q^m.$$

Let $h^k_{q,w}(t) = \sum_{n=0}^{\infty} G^{(k-1,k)}_{n,q}\frac{t^n}{n!}$. Then we have

$$h^k_{q,w}(t) = \sum_{n=0}^{\infty} G^{(k-1,k)}_{n+k,q}\frac{t^{n+k}}{(n+k)!} = [2]_q^k t^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}_q(-w)^m e^{[m]_q t}.$$

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